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Propagation of water waves over an infinite step

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Theoretical and experimental results are presented for the reflexion and transmission of water waves, passing over a step bottom between regions of finite and infinite depth. Two-dimensional motion is assumed, with the wave crests parallel to the step, and in the theory linearized irrotational flow is assumed. By matching 'wavemaker' solutions for the two regions at the cut above the step, an integral equation is derived for the horizontal velocity component on the cut. This integral equation is solved numerically and the reflexion and transmission coefficients and associated phase shifts are obtained. These results are compared with the long-wave theory and significant frequency effects are found, even for quite long waves. Experimental results are presented, which are in fair agreement with the theory.

1. Introduction

Problems involving the propagation of water waves in a fluid of variable depth can be conveniently divided into three categories: 'beach' problems, where the depth tends to zero; 'obstacle' problems, where the depth is a constant except for variations extending over a finite interval in space; and 'changing-depth' problems, where the depth changes from one limiting (non-zero) value to a second limiting (non-zero) value. There have been many investigations of the beach and obstacle problems (c.f. Stoker 1957, or Wehausen & Laitone 1960), but comparatively few studies have been made of the 'changing-depth' case. Much of the theoretical ground-work exists in the work of Kreisel (1949), but the only detailed studies are limited to very special geometries (Roseau 1952), or are based on the long-wave approximation (Bartholomeusz 1958 and Sretenskii 1950). The importance of wave propagation in the case of changing depth is obvious in many coastal situations, such as the passage of waves over a continental shelf. As an idealization of such a problem, we consider here the case of wave propagation over an infinite step, with constant finite depth on one side of the step and infinite depth on the other (figure 1). This is the situation treated by Sretenskii (1950) for oblique waves, but Sretenskii assumes that the wavelength is large compared to the finite depth and the results are not completely consistent.

The present study is in fact more closely related to that of Bartholomeusz (1958), which made more rigorous the solution of Lamb (1932) for long waves passing over a finite step, from one constant depth to another. Our situation is different in so far as one depth, rather than the wavelength, is considered infinitely large, but it will be shown that in the mutual limit of the two problems, i.e. long waves and one infinite depth, our results are consistent. This conclusion suggests that in the Lamb–Bartholomeusz problem the wavelength need only be long compared with the shallower of the two depths. An obvious further generalization would be to consider the finite-step geometry for finite wavelengths, and in fact the necessary analysis is essentially identical with that given in the present work.

One treatment of the obstacle problem which is closely related to our study is that of Jolas (1960), who considered the reflexion and transmission of waves incident upon a submerged rectangular parallelepiped. While it might seem that the infinite step is a special case of the rectangular parallelepiped, with the horizontal dimension tending to infinity, this is in fact an over-simplification since interference effects will persist between the two ends of the obstacle regardless of its length. However, these interference effects can be analysed, and the long obstacle can be treated as a synthesis of two steps placed 'back-to-back'. The necessary analysis is presented in a separate paper (Newman 1965).

Our treatment is based upon the usual assumptions of linearized water-wave theory and is restricted to the two-dimensional motion associated with wave crests parallel to the step. The fluid is assumed to be ideal, the motion irrotational, and the amplitude of the waves small compared with the wavelength and the fluid depth. There results a linear two-dimensional boundary-value problem for the velocity potential in the domain of the fluid. Assuming incident plane progressive waves of known amplitude, this boundary-value problem is reduced to an integral equation which is solved numerically for the reflexion and transmission coefficients. The analysis is performed for the case of waves incident from the deep fluid into the region of finite depth, but following Kreisel (1949) the reflexion and transmission coefficients for waves incident from the opposite direction (i.e. from the finite depth) can also be found from the relation existing between the two problems.

2. The boundary-value problem

Let (x, y) be Cartesian co-ordinates, with $y = 0$ the plane of the undisturbed free surface and y being positive downwards (figure 1). The fluid occupies the two regions

$$0 < y < h, \quad -\infty < x < 0;$$

and

$$0 < y < \infty, \quad 0 < x < \infty.$$

Assuming plane progressive waves and linearized theory throughout, the fluid-velocity vector may be represented by

$$\mathbf{V} = \mathrm{Re}[e^{-i\sigma t}\nabla\phi(x,y)],$$

where $\phi(x, y)$ is the velocity potential. This potential must satisfy the Laplace equation throughout the fluid domain and the following boundary conditions (Wehausen & Laitone 1960):

$$K\phi + \partial\phi/\partial y = 0 \quad \text{on} \quad y = 0, -\infty < x < \infty, \tag{2.1}$$

$$\partial\phi/\partial y = 0 \quad \text{on} \quad y = h, -\infty < x < 0, \tag{2.2}$$

$$\partial\phi/\partial x = 0 \quad \text{on} \quad x = 0, h < y < \infty, \tag{2.3}$$

and for $x > 0$ ϕ must be bounded as $y \to \infty$. The first boundary condition is the free surface condition and $K = \sigma^2/g$ is the wave-number for plane waves in a fluid of infinite depth. In addition, a radiation condition must be imposed. We

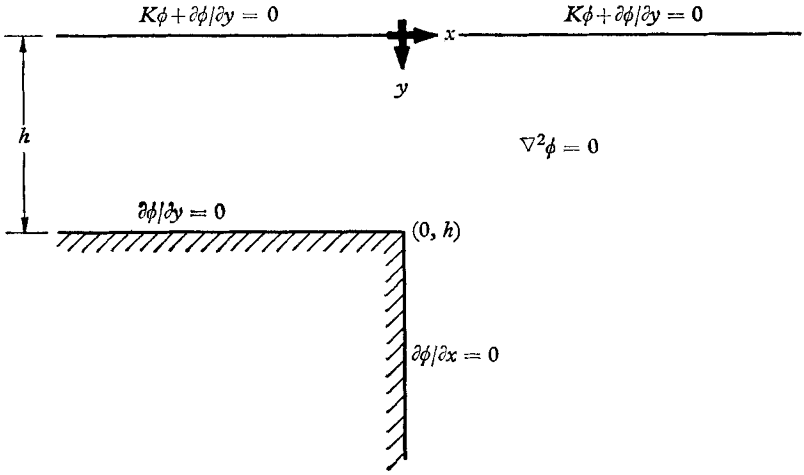

FIGURE 1. The boundary-value problem for the infinite step bottom.

shall consider, for the time being, the case of incident waves from the deep-water side, which are partially transmitted into the shallow water and partially reflected back into the deep water. The radiation condition is then

$$\phi \to \phi_I \, e^{-Ky-iKx} + \phi_R \, e^{-Ky+iKx} \quad \text{as} \quad x \to +\infty,$$

and
$$\phi \to \phi_T \cosh K_0(y-h) \, e^{-iK_0 x} \quad \text{as} \quad x \to -\infty,$$

where ϕ_I, ϕ_R, and ϕ_T are complex constants, and K_0 is the real positive root of the equation

$$K = K_0 \tanh K_0 h.$$

The above conditions state that as $x \to +\infty$ the motion consists of an incoming incident wave of amplitude $(\sigma/g)\,|\phi_I|$ plus an outgoing reflected wave of amplitude $(\sigma/g)\,|\phi_R|$, while as $x \to -\infty$ the motion consists of an outgoing transmitted wave of amplitude $(\sigma/g)\,|\phi_T| \cosh K_0 h$. The coefficient ϕ_I is assumed to be known, while ϕ_R and ϕ_T follow from the solution of the problem. In particular, the reflexion and transmission coefficients are defined as

$$R = \phi_R/\phi_I,$$

and
$$T = (\phi_T/\phi_I) \cosh K_0 h.$$

3. Derivation of the integral equation

To solve the problem stated in the preceding section we shall employ the 'wavemaker' theory of Havelock (1929), matching solutions valid for $x > 0$ and $x < 0$ at the 'cut' $x = 0$, $0 < y < h$. Specifically, we assume that on this line the horizontal velocity $\partial\phi/\partial x = U(y)$ is a known complex function. Then the potential on each side of the cut can be found immediately from the wavemaker theory, in terms of the function $U(y)$, and by matching these two potentials on the cut an integral equation is obtained for the function $U(y)$.

Consider first the region $x \geqslant 0$, $0 < y < \infty$, and let

$$\phi(x,y) = \phi_I e^{-Ky-iKx} + \psi(x,y). \tag{3.1}$$

The new potential ψ behaves like an outgoing wave at $x = +\infty$ and it satisfies the free surface condition (2.1). On $x = 0$ we have the boundary condition

$$\partial\psi/\partial x = U(y) + iK\phi_I e^{-Ky} \quad \text{for} \quad 0 < y < \infty,$$

and from (2.3) $\qquad U(y) = 0 \quad \text{for} \quad h < y < \infty.$

From Havelock's wavemaker theory it follows that, for $x > 0$,

$$\begin{aligned}\psi(x,y) = &-2i\,e^{-Ky+iKx}\int_0^\infty [U(\eta)+iK\phi_I e^{-K\eta}]\,e^{-K\eta}\,d\eta \\ &-\frac{2}{\pi}\int_0^\infty [U(\eta)+iK\phi_I e^{-K\eta}]\int_0^\infty \frac{e^{-kx}}{k(k^2+K^2)} \\ &\qquad\times(k\cos ky - K\sin ky)\,(k\cos k\eta - K\sin k\eta)\,dk\,d\eta \\ = &\;\phi_I e^{-Ky+iKx} - 2i\,e^{-Ky+iKx}\int_0^h U(\eta)\,e^{-K\eta}\,d\eta \\ &-\frac{2}{\pi}\int_0^h U(\eta)\int_0^\infty \frac{e^{-kx}}{k(k^2+K^2)}(k\cos ky - K\sin ky)\,(k\cos k\eta - K\sin k\eta)\,dk\,d\eta.\end{aligned} \tag{3.2}$$

Subsituting (3.2) in (3.1) and setting $x = 0+$, we obtain

$$\begin{aligned}\phi_+ \equiv \phi(0+,y) = &\;2\phi_I e^{-Ky} - 2i\,e^{-Ky}\int_0^h U(\eta)\,e^{-K\eta}\,d\eta \\ &-\frac{2}{\pi}\int_0^h U(\eta)\int_0^\infty \frac{1}{k(k^2+K^2)}(k\cos ky - K\sin ky)\,(k\cos k\eta - K\sin k\eta)\,dk\,d\eta.\end{aligned} \tag{3.3}$$

Proceeding in an analogous manner for $x < 0$, the potential ϕ behaves like an outgoing wave at $x = -\infty$; it satisfies the boundary conditions (2.1) and (2.2) on the free surface and on the bottom, respectively, and on $x = 0$ the boundary condition

$$\partial\phi/\partial x = U(y). \tag{3.4}$$

Thus from the wavemaker theory for finite depth, the velocity potential for the region $x < 0$ is

$$\begin{aligned}\phi(x,y) = &\;\frac{2i\,e^{-iK_0 x}\cosh K_0(y-h)}{K_0 h + \frac{1}{2}\sinh 2K_0 h}\int_0^h U(\eta)\cosh K_0(\eta-h)\,d\eta \\ &+\sum_{n=1}^\infty \frac{2e^{k_n x}\cos k_n(y-h)}{k_n h + \frac{1}{2}\sin 2k_n h}\int_0^h U(\eta)\cos k_n(\eta-h)\,d\eta,\end{aligned} \tag{3.5}$$

where k_n is the nth positive real root of the equation

$$K + k_n \tan k_n h = 0.$$

The validity of (3.5) is easily checked, for each term separately is a potential function satisfying the free surface and bottom conditions; the first term itself satisfies the radiation condition while the remaining terms are exponentially small at $x = -\infty$; and, finally, the boundary condition (3.4) can be verified by differentiating (3.5), setting $x = 0-$, and noting that the infinite set of functions

$$\{\cosh K_0(y-h),\ \cos k_n(y-h)\}$$

is complete and orthogonal in the interval $0 < y < h$. Setting $x = 0-$ in (3.5) we obtain

$$\phi_- \equiv \phi(0-, y) = \frac{2i \cosh K_0(y-h)}{K_0 h + \frac{1}{2}\sinh 2K_0 h} \int_0^h U(\eta) \cosh K_0(\eta - h)\, d\eta$$
$$+ \sum_{n=1}^{\infty} \frac{2\cos k_n(y-h)}{k_n h + \frac{1}{2}\sin 2k_n h} \int_0^h U(\eta) \cos k_n(\eta - h)\, d\eta. \quad (3.6)$$

We now proceed to match the two solutions at the cut. For this purpose it is required that the velocity be continuous across the cut. We have already ensured that the horizontal component is continuous since $\partial\phi/\partial x = U(y)$ on both sides of the cut, but there remains the condition on the vertical velocity,

$$\frac{\partial}{\partial y}(\phi_+ - \phi_-) = 0, \quad \text{on} \quad 0 < y < h.$$

Then ϕ_+ and ϕ_- can differ only by a constant, and from the free surface condition this constant must be zero. Thus our matching condition is

$$\phi_+ = \phi_- \quad \text{on} \quad 0 < y < h.$$

Equating (3.3) and (3.6) we then obtain the desired integral equation for $U(y)$:

$$\int_0^h U(\eta)\, L(y, \eta)\, d\eta = \phi_I e^{-Ky}, \quad (3.7)$$

where the kernel $L(y, \eta)$ is defined by

$$L(y,\eta) = i\, e^{-K(y+\eta)} + \frac{i \cosh K_0(y-h) \cosh K_0(\eta - h)}{K_0 h + \frac{1}{2}\sinh 2K_0 h}$$
$$+ \frac{1}{\pi}\int_0^\infty \frac{1}{k(k^2+K^2)} (k \cos ky - K \sin ky)(k \cos k\eta - K \sin k\eta)\, dk$$
$$+ \sum_{n=1}^{\infty} \frac{\cos k_n(y-h) \cos k_n(\eta - h)}{k_n h + \frac{1}{2}\sin 2k_n h}. \quad (3.8)$$

The integral over k can be evaluated from known Fourier transforms, in terms of the logarithmic function and the exponential integral

$$\overline{Ei}(x) = \int_{-\infty}^{x} e^t\, dt/t.$$

Thus the kernel can be written in the form

$$L(y,\eta) = i\, e^{-K(y+\eta)} + \frac{i \cosh K_0(y-h) \cosh K_0(\eta - h)}{K_0 h + \frac{1}{2}\sinh 2K_0 h}$$
$$+ \frac{1}{2\pi}\log\left|\frac{y+\eta}{y-\eta}\right| - \frac{1}{\pi} e^{-K(y+\eta)}\overline{Ei}(Ky + K\eta)$$
$$+ \sum_{n=1}^{\infty} \frac{\cos k_n(y-h) \cos k_n(\eta - h)}{k_n h + \frac{1}{2}\sin 2k_n h}. \quad (3.9)$$

Similarly, the infinite series can be expressed by a logarithmic function plus a principal-value integral, following Bartholomeusz (1958), but for the present purposes there is no advantage in doing so.

Bartholomeusz has derived the more general integral equation corresponding to a finite step, and the kernel (3.9) may be regarded as a special case of Bartholomeusz's kernel, in the limit of infinite depth on one side. Since (3.7) is an integral equation of the first kind it is not immediately obvious that a unique solution exists, but in fact the necessary proof has been obtained, by transforming (3.7) into a regular Fredholm equation of the second kind, the solution of which exists and is unique at least for sufficiently small values of Kh. The details follow directly from the corresponding proof of Bartholomeusz, and are not considered to be of sufficient interest to include here.

4. Numerical solution of the integral equation

In order to find numerical solutions of the integral equation (3.7) we expand the unknown function $U(\eta)$ in terms of the complete orthogonal set of functions corresponding to the wavemaker theory in finite depth. Thus

$$U(\eta) = \sum_{n=0}^{\infty} U_n \cos k_n(\eta - h) \quad \text{for} \quad 0 \leqslant \eta \leqslant h, \tag{4.1}$$

where the unknown coefficients U_n are in general complex, and where for compactness we have introduced the new constant

$$k_0 = iK_0. \tag{4.2}$$

Substituting the expansion (4.1) in the integral equation (3.7) and interchanging the orders of summation and integration, we obtain

$$\sum_{n=0}^{\infty} U_n \int_0^h \cos k_n(\eta - h) L(y,\eta)\, d\eta = \phi_I e^{-Ky}. \tag{4.3}$$

Multiplying both sides by $\cos k_m(y-h)$ and integrating over the interval $0 < y < h$, we then obtain the infinite system of simultaneous equations

$$\sum_{n=0}^{\infty} U_n \int_0^h \int_0^h \cos k_m(y-h) \cos k_n(y-h) L(y,\eta)\, d\eta\, dy$$
$$= \phi_I \int_0^h e^{-Ky} \cos k_m(y-h)\, dy \quad (m = 0, 1, 2, \ldots). \tag{4.4}$$

The integrals in (4.4) can all be evaluated, in terms of elementary functions or sine, cosine, and exponential integrals. It can be shown that

$$\int_0^h e^{-Ky} \cos k_m(y-h)\, dy = -\frac{K e^{-Kh}}{K^2 + k_m^2},$$

while, for $m \neq n$,

$$\int_0^h \int_0^h \cos k_m(y-h) \cos k_n(\eta - h) L(y,\eta)\, d\eta\, dy = \frac{K^2 e^{-2Kh}[\pi i - \overline{Ei}(2Kh)]}{\pi(K^2 + k_m^2)(K^2 + k_n^2)}$$
$$+ \frac{1}{2\pi(k_m^2 - k_n^2)} [\log(k_n/k_m) + Ci(2k_m h) - Ci(2k_n h)],$$

and, for $m = n$,

$$\int_0^h \int_0^h \cos k_m(y-h) \cos k_m(\eta - h)\, L(y,\eta)\, d\eta\, dy$$

$$= \frac{K^2 e^{-2Kh}[\pi i - \overline{Ei}(2Kh)]}{\pi(K^2+k_m^2)^2} + \frac{(K^2h + k_m^2 h - K)}{2\pi k_m(K^2+k_m^2)} Si(2k_m h)$$

$$- \frac{K^2}{2\pi k_m^2(K^2+k_m^2)} + \frac{k_m(K^2h + k_m^2 h - K)}{4\,|k_m|^2\,(K^2+k_m^2)}.$$

Here, in the usual notation,

$$\overline{Ei}(x) = \int_{-\infty}^{x} e^t dt/t, \quad Si(x) = \int_0^x \sin t\, dt/t \quad \text{and} \quad Ci(x) = -\int_x^\infty \cos t\, dt/t.$$

This system of equations (4.4) is complex, and thus represents two coupled systems of real equations. These can be uncoupled by straightforward linear transformations of the unknown coefficients, but the algebraic details are rather cumbersome and will not be pursued here. The resulting pair of uncoupled systems has been solved numerically by truncation and standard methods for solving a finite system of equations, using the I.B.M. 7090 digital computer. In order to ensure satisfactory convergence of this system the computations were repeated for each value of Kh, with successively 10, 20, 40, and 80 terms and equations retained. In the range of Kh between 0·001 and 4 the maximum deviation observed was 0·0002 in the magnitude of the reflexion and transmission coefficients. The arguments of these coefficients (or the phase angles) converged more slowly with a maximum deviation of 1·2° at the highest frequencies, but even there the maximum deviation was only 0·1° between 40 and 80 terms. The results of these computations are presented in the next section.

5. The reflexion and transmission coefficients

The reflexion and transmission coefficients may be found from the asymptotic behaviour of ϕ. From equation (3.2) the potential of the reflected wave at $x = +\infty$ is

$$\psi \simeq e^{-Ky+iKx}\left[\phi_I - 2i\int_0^h U(\eta)\, e^{-K\eta}\, d\eta\right] \equiv \phi_R\, e^{-Ky+iKx}.$$

Thus $$\phi_R = \phi_I - 2i\int_0^h U(\eta)\, e^{-K\eta}\, d\eta = \phi_I + 2iK\, e^{-Kh} \sum_{n=0}^{\infty} \frac{U_n}{K^2+k_n^2},$$

and the magnitude of the reflexion coefficient is given by

$$|R| = |\phi_R/\phi_I| = \left|1 - \frac{2i}{\phi_I}\int_0^h U(\eta)\, e^{-K\eta}\, d\eta\right| = \left|1 + \frac{2i}{\phi_I} K\, e^{-Kh} \sum_{n=0}^{\infty} \frac{U_n}{K^2+k_n^2}\right|. \quad (5.1)$$

From equation (3.5) the transmitted wave at $x = -\infty$ is

$$\phi \simeq \frac{2i\, e^{-iK_0 x} \cosh K_0(y-h)}{K_0 h + \frac{1}{2}\sinh 2K_0 h} \int_0^h U(\eta) \cosh K_0(\eta - h)\, d\eta$$

$$= \frac{iU_0}{K_0} e^{-iK_0 x} \cosh K_0(y-h) \equiv \phi_T\, e^{-iK_0 x} \cosh K_0(y-h). \quad (5.2)$$

Thus $$\phi_T = iU_0/K_0,$$

and the magnitude of the transmission coefficient is

$$|T| = |\phi_T/\phi_I| \cosh K_0 h = |U_0/\phi_I| \frac{\cosh K_0 h}{K_0}. \tag{5.3}$$

In a similar manner we can obtain the phase shifts

$$\arg(R) \equiv \delta R = \arg(\phi_R/\phi_I) = \arg\left(1 + \frac{2i}{\phi_I} K e^{-Kh} \sum_{n=0}^{\infty} \frac{U_n}{K^2 + k_n^2}\right) \tag{5.4}$$

and

$$\arg(T) \equiv \delta T = \arg(\phi_T/\phi_I) = \arg(iU_0/\phi_I). \tag{5.5}$$

The above relations apply to the case of incident waves from the deep region $x \to +\infty$. Conversely, we can consider the analogous problem with incident waves from the shallow region $x \to -\infty$. Following Kreisel (1949), there exist relations

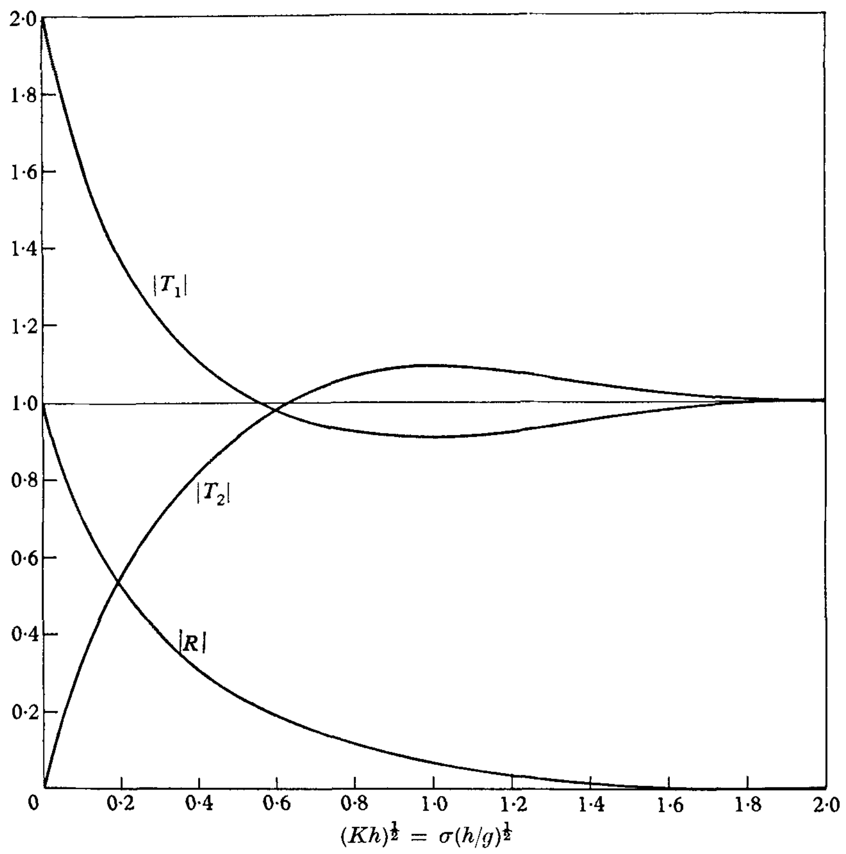

FIGURE 2. Reflexion and transmission coefficients.

$$|T_1| = \frac{\text{Amplitude of wave transmitted into shallow water}}{\text{Amplitude of wave incident from deep water}},$$

$$|T_2| = \frac{\text{Amplitude of wave transmitted into deep water}}{\text{Amplitude of wave incident from shallow water}},$$

$$|R| = \frac{\text{Amplitude of reflected wave}}{\text{Amplitude of incident wave}} \quad (|R_1| = |R_2| \equiv |R|).$$

between the two problems of the following form; let us denote the two above cases by subscripts one and two, respectively. Then Kreisel has shown that

$$|R_1| = |R_2| \equiv |R|. \tag{5.6}$$

Similarly, it can be shown (Newman 1965) that

$$|T_1 T_2| = 1-|R|^2, \tag{5.7}$$

$$\delta T_1 = \delta T_2 \equiv \delta T, \tag{5.8}$$

and

$$\delta R_1 + \delta R_2 = \pi + 2\delta T. \tag{5.9}$$

The three coefficients $|R|$, $|T_1|$ and $|T_2|$ are shown in figure 2, as functions of the non-dimensional frequency parameter $\sigma(h/g)^{\frac{1}{2}} = (Kh)^{\frac{1}{2}}$. The corresponding phase angles δR_1, δR_2 and δT are shown in figure 3.

One striking feature of figure 2 is the limit $|T_1| \to 2$ as $Kh \to 0$. Thus for sufficiently small frequencies (or shallow depths) the height of the transmitted

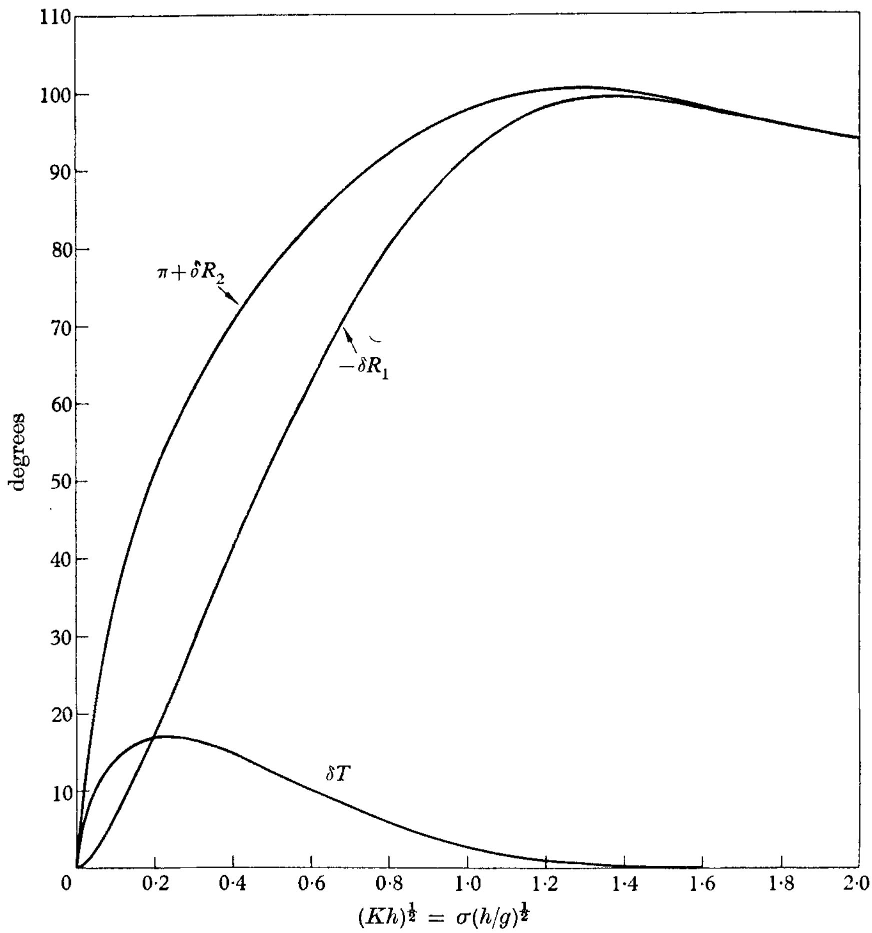

FIGURE 3. Phase shifts for transmission and reflexion. δT = phase of transmitted wave less phase of incident wave ($\delta T_1 = \delta T_2 \equiv \delta T$); δR_1 = phase of reflected wave less phase of incident wave in deep water; δR_2 = phase of reflected wave less phase of incident wave in shallow water.

wave in the shallow region approaches a value precisely twice that of the incident wave. This result is explained by examining, in the following section, the asymptotic behaviour of the solution in the limit of low frequency or long waves.

6. The long-wave limit

The asymptotic behaviour in the limit of long waves, or small Kh, can be obtained either from the integral equation (3.7) or from the simultaneous equations (4.4). Utilizing the integral equation, the limit of the kernel (3.9) for small K is

$$L(y,\eta) = \frac{i}{2K_0 h} + i + \frac{1}{2\pi}\log\left|\frac{y+\eta}{y-\eta}\right| - \frac{1}{\pi}[\gamma + \log(Ky+K\eta)]$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\pi n}\cos \pi n(1-y/h)\cos \pi n(1-\eta/h) + O([Kh]^{\frac{1}{2}}), \quad (6.1)$$

where we have used the facts that

$$K_0 h = (Kh)^{\frac{1}{2}} + O([Kh]^{\frac{3}{2}}), \quad k_n h = \pi n + O(Kh),$$

and

$$\overline{Ei}(x) = \gamma + \log x + O(x), \quad \text{as} \quad x \to 0.$$

Here $\gamma = 0{\cdot}577\ldots$ is the Euler–Mascheroni constant. The value of the infinite series in equation (6.1) is

$$-(2\pi)^{-1}\log|2\cos(\pi y/h) - 2\cos(\pi\eta/h)|.$$

Thus

$$L(y,\eta) = (2\pi)^{-1}\{\pi i(K_0 h)^{-1} + 2\pi i - 2\log Kh - 2\gamma$$
$$-\log|2\cos(\pi y/h) - 2\cos(\pi\eta/h)| - \log|(y^2/h^2)-(\eta^2/h^2)|\} + O(K^{\frac{1}{2}}h^{\frac{1}{2}}). \quad (6.2)$$

Substituting in (3.7), we obtain the integral equation

$$i(2K_0h)^{-1}(1+2K_0h+2i\pi^{-1}K_0h\log Kh)\int_0^h U(\eta)\,d\eta + \int_0^h U(\eta)\,R(y,\eta)\,d\eta \simeq \phi_I, \quad (6.3)$$

where $R(y,\eta)$ denotes the real kernel

$$R(y,\eta) = (2\pi)^{-1}\{-2\gamma - \log|2\cos(\pi y/h) - 2\cos(\pi\eta/h)| - \log|(y^2/h^2)-(\eta^2/h^2)|\},$$

and the error in (6.3) is a factor $1+O(Kh)$. The solution of this integral equation is

$$U(\eta) = \left[\phi_I - \frac{1}{2K_0 h}(1+2K_0h+\frac{2i}{\pi}K_0h\log Kh)\int_0^h U(\eta)\,d\eta\right]u(\eta), \quad (6.4)$$

where $u(\eta)$ is the (real) solution of the normalized integral equation†

$$\int_0^h u(\eta)\,R(y,\eta) = 1.$$

† It can be shown that this solution exists and is unique; see the last paragraph of § 3.

Integrating both sides of (6.4), we find that

$$\int_0^h U(\eta)\,d\eta = -2iK_0 h\phi_I\left[1+2K_0h+2i\pi^{-1}K_0h\log Kh-2iK_0h\bigg/\int_0^h u(\eta)\,d\eta\right]^{-1}$$

$$= -2iK_0 h\phi_I\left[1-2K_0h-2i\pi^{-1}K_0h\log Kh+2iK_0h\bigg/\int_0^h u(\eta)\,d\eta\right] + O(K_0^3h^3\log^2 Kh).$$

The constant $$\int_0^h u(\eta)\,d\eta$$

can be computed since it is the flux associated with a streaming flow past the step, with a rigid free surface. However, this analysis is not necessary for, since $u(\eta)$ is real, it follows directly that

$$\left|\frac{1}{\phi_I}\int_0^h U(\eta)\,d\eta\right| = 2K_0h-4(K_0h)^2-\frac{4}{\pi}(K_0h)^3\log^2 Kh+O(K_0^3h^3\log Kh).$$

Expanding (5.2) and (5.3) for small K_0h, the magnitude of the transmission coefficient is obtained as

$$\begin{aligned}|T_1| &= |\phi_T/\phi_I|\cosh K_0h\\ &= \frac{1}{K_0h}\left|\frac{1}{\phi_I}\int_0^h U(\eta)\,d\eta\right|+O(Kh)\\ &= 2-4(Kh)^{\frac{1}{2}}-\frac{4}{\pi}Kh\log^2 Kh+O(Kh\log Kh). \qquad (6.5)\end{aligned}$$

The magnitude of the reflexion coefficient is obtained in a similar manner, using (5.1), with the result that

$$|R| = 1-4(Kh)^{\frac{1}{2}}+8Kh+O(K^{\frac{3}{2}}h^{\frac{3}{2}}\log^2 Kh). \qquad (6.6)$$

Finally, from (5.7)

$$|T_2| = \frac{1-|R|^2}{|T_1|} = 4(Kh)^{\frac{1}{2}}-8Kh+O(K^{\frac{3}{2}}h^{\frac{3}{2}}\log^2 Kh). \qquad (6.7)$$

The asymptotic approximations (6.5)–(6.7) are in good agreement with the numerical results shown in figure 2. That $|R| \to 1$ as $Kh \to 0$ is not unexpected, since the step is tending in the limit to a wall which will be totally reflecting. Since the two reflexion coefficients for waves incident from the deep and shallow sides of the step are equal, it follows that, as $Kh \to 0$, waves incident from the shallow fluid will also be totally reflected with no energy radiation into the deep fluid. The limit $T_2 \to 0$ is an obvious consequence, but the limit $|T_1| \to 2$ is more difficult to understand physically; apparently as $Kh \to 0$ the energy transmitted into the shallow fluid vanishes at a rate proportional to the depth h, such that the transmitted wave is just twice the height of the incident wave.

The above limits are consistent with the long-wave theory of Lamb (1932) for the case of a finite step, which has been verified by Bartholomeusz (1958). In this theory, if the two depths are h_1 and h_2 and both are small compared with the wavelength, then, for waves incident from the depth h_1, the transmission and reflexion coefficients are

$$|T| = 2(h_1/h_2)^{\frac{1}{2}}/|1-(h_1/h_2)^{\frac{1}{2}}|, \tag{6.8}$$

and

$$|R| = [1+(h_1/h_2)^{\frac{1}{2}}]/|1-(h_1/h_2)^{\frac{1}{2}}|. \tag{6.9}$$

Letting $h_1 \to \infty$ we obtain the limits $|T_1| = 2$ and $|R| = 1$, while for $h_2 \to \infty$ we obtain the corresponding limits $|T_2| = 0$ and $|R| = 1$.

7. Experimental verification

An experimental investigation was performed to verify the theoretical results obtained above. For this purpose use was made of a small towing tank, approximately 15 m in length, 60 cm wide, and 60 cm deep. This tank was equipped with a pneumatic wavemaker at one end, capable of generating sinusoidal waves of approximately 20 cm to 10 m in wavelength. At the opposite end of the tank a step bottom was inserted, consisting of an aluminium platform 6 m long with end plates 50 cm high, extending across the tank and down to the bottom. The depth of fluid above this platform was varied from 3·7 cm to 15 cm by adjusting the water level. Thus in fact the experimental set-up corresponded to a finite-step bottom, from a depth of about 60 cm to a depth of a few cm, but for small and moderate wavelengths the 60 cm depth is essentially infinite. At the far end of the platform a beach was placed to absorb the transmitted waves. This beach consisted of several layers of steel mesh screen, 180 cm in length and sloping up from the platform to the free surface.

Wave-height measurements were made with a sonic-type wave probe, which has been described by Killen (1962). Briefly this probe consists of a spark gap and capacitance microphone, situated about 15 cm above the free surface. The spark gap is excited by a pulse at a frequency of 120 cycles/sec, and the microphone responds to the resulting sound generated by the spark after being reflected from the free surface. By electronically timing the interval between the generation of the pulse and the arrival of the reflected sound wave, a measure of the distance to the free surface is obtained, and hence the free surface elevation can be found. By amplifying and filtering the resulting signal, and recording this on a standard pen recorder, waves of about 1 cm amplitude can be measured with a resolution of approximately 0·1 mm. This sensitivity was required to avoid the non-linear effects associated with higher waves in the shallow water.

In order to resolve the reflected and transmitted waves, a technique was employed similar to that of Dean & Ursell (1959). Assuming the deep-water wave to consist of an incident wave and a reflected wave, the wave observed at different points along the tank will vary in amplitude between a maximum, where the two components are in phase, and a minimum, where they are out of phase. The maximum amplitude observed will thus equal the sum of the two wave components, or

$$|\eta_{\max}| = |\eta_I| + |\eta_R|,$$

while the minimum amplitude observed will be equal to their difference,

$$|\eta_{\min}| = |\eta_I| - |\eta_R|.$$

Thus

$$|\eta_I| = \tfrac{1}{2}(|\eta_{\max}| + |\eta_{\min}|),$$

and

$$|\eta_R| = \tfrac{1}{2}(|\eta_{\max}| - |\eta_{\min}|),$$

and the magnitude of the reflexion coefficient is given by

$$|R| = \frac{|\eta_{max}| - |\eta_{min}|}{|\eta_{max}| + |\eta_{min}|}. \tag{7.1}$$

Similarly, if we use the symbol ζ for the shallow-water-wave amplitudes, and allow for the presence of beach reflexions, the transmitted amplitude is

$$|\eta_T| = \tfrac{1}{2}(|\zeta_{max}| + |\zeta_{min}|),$$

and thus the magnitude of the transmission coefficient is

$$|T_1| = \frac{|\zeta_{max}| + |\zeta_{min}|}{|\eta_{max}| + |\eta_{min}|}. \tag{7.2}$$

Moreover, the reflexion coefficient for the beach is

$$|R_B| = \frac{|\zeta_{max}| - |\zeta_{min}|}{|\zeta_{max}| + |\zeta_{min}|}. \tag{7.3}$$

The procedure adopted consisted of mounting the probe on a towing carriage, which was moved up and down the length of the tank at a very slow speed (about 3 cm/sec). Generally the probe was started near the wavemaker end, in the deep-water section, after the wave motion had reached a steady state in this region. As the probe moved up through the deep section, measurements of $|\eta_{max}|$ and $|\eta_{min}|$ could be obtained; in the vicinity of the step, local effects existed and this portion of the record was ignored; then, after passing across the step into the shallow water region, measurements of $|\zeta_{max}|$ and $|\zeta_{min}|$ could be found. Generally when the probe reached the vicinity of the beach, its motion was reversed and a second set of data was obtained from the return trip. In this way two sets of independent data were obtained from each run, and their consistency provided an estimate of the steady-state nature of the waves. One of the advantages of this wave-measuring system is that only one probe is required, and, since the reflexion and transmission coefficients involve only ratios of its signal, no accurate calibration need be performed.

The above procedure is straightforward but does not account for the effect of beach reflexions on the measured coefficients. The measured beach reflexion coefficient $|R_B|$ varied from 0·03 to 0·26, with most values of $|R_B|$ in the range 0·10 to 0·18. Since the reflexion coefficient from the step is of the same order, it was necessary to correct for beach reflexions when these were measurable. The effect of beach reflexions can be analysed in the following way, which is basically similar to the 'reversed-time' method of Dean & Ursell (1959). In the notation of Kreisel, the measured wave including beach reflexions is

$$\phi_0 = \{A_0, B_0;\ a_0, b_0\}.$$

Here A and B denote the complex amplitude of the incident and reflected waves in the deep region, and a and b denote the complex amplitudes of the transmitted and reflected waves in the shallow region. Thus $|R_B| = |b_0/a_0|$ is the beach reflexion coefficient. The desired potential

$$\phi_1 = \{A_1, B_1;\ a_1, 0\}$$

is free from beach reflexions, and the desired reflexion and transmission coefficients of the step are

$$|R| = |B_1/A_1|$$

and

$$|T_1| = |a_1/A_1| \cosh K_0 h.$$

The potential ϕ_1 can be constructed by superposition of the measured wave and its conjugate,

$$\phi_1 = \phi_0 - (b_0/\bar{a}_0)\,\bar{\phi}_0,$$

so that

$$A_1 = A_0 - b_0\bar{B}_0/\bar{a}_0, \quad B_1 = B_0 - b_0\bar{A}_0/\bar{a}_0, \quad a_1 = a_0 - b_0\bar{b}_0/\bar{a}_0,$$

and the desired reflexion and transmission coefficients are

$$|R| = \left|\frac{B_0\bar{a}_0 - b_0\bar{A}_0}{A_0\bar{a}_0 - b_0\bar{B}_0}\right|, \tag{7.4}$$

and

$$|T_1| = \left|\frac{a_0\bar{a}_0 - b_0\bar{b}_0}{A_0\bar{a}_0 - b_0\bar{B}_0}\right| \cosh K_0 h. \tag{7.5}$$

If we define the measured coefficients

$$|R_M| = |B_0/A_0|, \quad |T_M| = |a_0/A_0| \cosh K_0 h, \quad \text{and} \quad |R_B| = |b_0/a_0|,$$

and recall that

$$\delta T = \arg(a_1/A_1),$$

then after some straightforward reduction we obtain the relations

$$|R| = |R_M| - (1 - |R_M|^2)\,|R_B| \cos(\Theta - \theta + 2\delta T) + O(|R_B|^2) \tag{7.6}$$

and

$$|T_1| = T_M[1 + |R_M R_B| \cos(\Theta - \theta + 2\delta T)] + O(|R_B|^2), \tag{7.7}$$

where $\Theta = \arg(A_0/B_0)$ and $\theta = \arg(a_0/b_0)$. The two angles Θ and θ are measurable from the locations of the maximum and minimum observed wave heights. In the deep section, maxima occur at points x satisfying the relation

$$2Kx = 2\pi n + \Theta,$$

while minima occur at points x satisfying the relation

$$2Kx = 2\pi(n + \tfrac{1}{2}) + \Theta.$$

Similar relations hold for θ in shallow water, with the wave-number K_0 replacing K. In analysing the experimental records several such values of x were read and fitted to the above relations by a least-squares technique to find the phase angles Θ and θ. The theoretical value of δT was then used to supply the remaining information necessary to correct the coefficients R and T_1 for beach reflexions. This process was carried out for all runs where a clean signal was obtained in shallow water and judged to contain significant beach reflexions. It should be emphasized that the experimental determination of the phase angles Θ and θ is subject to the usual errors occurring in the measurement of phase angles, and this is especially true here, since the phase being measured is that of the amplitude modulation, rather than the signal itself.

The experimental values of $|R|$ and $|T_1|$ are shown in figure 4, along with the theoretical curves. With one exception the experimental points are shown in pairs, corresponding to the two directions of travel of the probe in each run. The scatter between each pair of points is a measure of the internal consistency of each run. The four different depths are shown by different symbols, and double symbols superposed denote data points for which a beach-reflexion correction

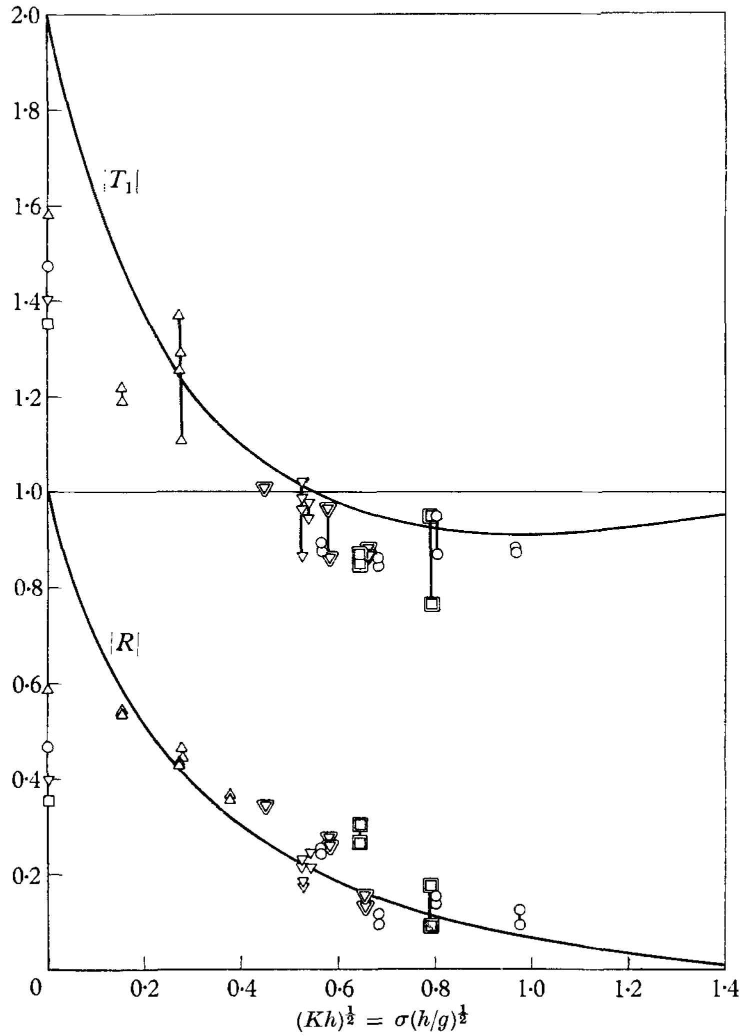

FIGURE 4. Comparison of experimental points with theoretical curves for reflexion coefficient $|R|$ and transmission coefficient $|T_1|$. Different symbols denote different fluid depths and double symbols superposed denote points corrected for beach reflexion. Long-wave limits for each depth, according to the theory of Lamb and Bartholomeusz, are indicated by the corresponding symbols located on the ordinate. Points joined by vertical lines denote pairs of data taken during the two directions of travel of the probe in each run. △, h = 3·8 cm; ○, h = 7·6 cm; ▽, h = 11·4 cm; □, h = 15·2 cm.

has been made. Also shown in figure 4, by the corresponding symbols situated on the ordinate, are the long-wave limits obtained from equations (6.8) and (6.9) including the effect of finite depth of the deep fluid; these theoretical limit-points help in interpreting the experimental behaviour at low frequencies. Generally speaking, the agreement between theory and experiments is moderately good, with the most serious discrepancies in the extreme conditions of long waves or large depth. The most likely sources of experimental error are beach reflexions, finite depth in the deep section, and physical irregularities in the tank and aluminium false bottom. Regarding the last, the width of the tank varied by about 7 mm, with a corresponding gap between the walls and the platform. Also the level of the platform, and thus the shallow depth, varied by about 3 mm. Viscous effects appeared insignificant, as dye pellets confirmed the existence of a thin boundary-layer and little appearance of vorticity, even near the edge of the step. Non-linear effects were checked by variation of the incident-wave height, with no significant results, as long as steep waves in the shallow water were avoided.

8. Conclusions

We have obtained theoretical values of the reflexion and transmission coefficients, including the corresponding phase shifts, associated with propagation of waves over a step-shaped bottom where the depth on the deep side of the step is infinite. In the limit of long wavelengths these results are consistent with the finite-step long-wave theory of Lamb (1932) and Bartholomeusz (1958), implying that in their analysis it is sufficient to require only that the wavelength be long compared with the lesser of the two depths. However, it is clear from our results that the long-wave limit is a poor approximation even at quite small values of the depth, for at $\omega(h/g)^{\frac{1}{2}} = 0{\cdot}2$, corresponding to a wavelength 50π or 157 times the depth of the shallow fluid, the reflexion coefficient is already reduced from 1·0 to 0·52, and the transmission coefficient $|T_1|$ is reduced from 2·0 to 1·37. Thus we may expect that, even for relatively long waves entering quite shallow water, frequency effects will be important.

This research was performed at the David Taylor Model Basin, with calculations performed on the I.B.M. 7090 digital computer of the Applied Mathematics Laboratory. Programming was performed by Mr W. Frank, while Messrs N. G. Milihram and B. L. Moore were especially helpful in carrying out the experimental phase. The author also wishes to express his gratitude to Drs W. E. Cummins and T. F. Ogilvie, and Mr V. J. Monacella, for their encouragement and support throughout the investigation, and to Dr Richard Holford for pointing out an error in the derivation of the long-wave limits.

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